

Numerical detection of symmetry-enriched topological phases with space-group symmetry

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Topologically ordered phases of matter, in particular so-called symmetry-enriched topological phases, can exhibit quantum number fractionalization in the presence of global symmetry. In \mathbb{Z}_2 topologically ordered states in two dimensions, fundamental translations T_x and T_y acting on anyons can either commute or anticommute. This property, crystal momentum fractionalization, can be seen in a periodicity of the excited-state spectrum in the Brillouin zone. We present a numerical method to detect the presence of this form of symmetry enrichment given a projected entangled pair state; we study the minima of the spectrum of correlation lengths of the transfer matrix for a cylinder. As a benchmark, we demonstrate our method using a modified toric code model with perturbation. An enhanced periodicity in momentum clearly reveals the nontrivial anticommutation relation $\{T_x, T_y\} = 0$ for the corresponding quasiparticles in the system.

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Topological order is the name given to a variety of long-range-entangled but gapped phases of matter, in particular to phases that support anyonic excitations with unusual braiding statistics. Unlike phases that fall within the Landau symmetry-breaking paradigm, topological order is defined without any reference to symmetry. However, it is still very interesting to ask what further phenomena emerge in systems with symmetry, either spontaneously broken or unbroken. The term “symmetry-enriched topological” (SET) phases was proposed to describe phases that have the same topological order but are distinct in the presence of a symmetry [1–3].

In two dimensions the excitations in a topological phase are pointlike anyons. When a symmetry is present, what are their quantum numbers? It turns out that these can be fractional; most prominently, anyons in quantum Hall states typically have fractional electric charge [the quantum number corresponding to a $U(1)$ symmetry of the system] [4,5]. The values of these quantum numbers are highly constrained, notably by the “fusion rules” of the topological theory; in the Laughlin quantum Hall state with filling fraction $1/3$, since three anyons give back an electron, the anyons must have charge $e/3$. Another example is the spin fractionalization in spin liquid phases of quantum magnets with a global $SU(2)$ symmetry: a spinon quasiparticle carries a fractional number spin- $1/2$ whereas in conventional paramagnets or magnetically ordered states all quasiparticles carry integer spin [6].

Can one distinguish SET phases given a wave function? The quantum numbers of the degenerate ground states and the projective quantum numbers of quasiparticles are the characteristic properties of the SET phases [2,7–9]. In the case of detecting the projective quantum number of an internal symmetry, the operation of the global internal symmetry generator can be factorized into a product of local operators, which can be transformed into operators acting only on the entanglement cut of the system; therefore one can detect the projective quantum number via measuring the commutator/anticommutator of the boundary quasiparticles [10,11]. However for the space-group symmetry, such as the translation symmetries, the translation operator is written in a matrix product operator that cannot be factorized, and in addition, there is no way to make an

entanglement cut that preserves both T_x and T_y translations; therefore conventional techniques do not work. In this Rapid Communication, we address this question for space-group symmetries in the context of projected entangled pair states (PEPSs) [12].

\mathbb{Z}_2 topological phases with translation symmetry. We are interested in the topological order familiar from \mathbb{Z}_2 gauge theory, \mathbb{Z}_2 spin liquids, and the toric code. There are two bosonic anyon species, often called e and m . Each sees the other with an Aharonov-Bohm phase of -1 (they are mutual semions). When two-dimensional translation symmetry is present, the symmetry generators T_x and T_y may act nontrivially (projectively) on the anyons [2,13]. A basis-independent characterization of these actions is given by the relations

$$\begin{aligned} T_x^e T_y^e T_x^{e-1} T_y^{e-1} &= \eta_e = \pm 1, \\ T_x^m T_y^m T_x^{m-1} T_y^{m-1} &= \eta_m = \pm 1, \end{aligned} \quad (1)$$

where T_x^e is the action of T_x on a single e , etc. When one of these relations evaluates to -1 we say that translations act projectively, or that the anyon has fractional crystal momentum or has nontrivial fractionalization class, a notion closely related to Wen’s projective symmetry group [14].

One can interpret the nontrivial e relation as the presence of an m in each unit cell of the lattice, which the e sees as a background π magnetic flux. It is straightforward to show, based on this observation or directly from the relations above, that if e has fractional crystal momentum, the spectrum (and density of states) of two- e scattering states is periodic under $\mathbf{q} \rightarrow \mathbf{q} + (\pi, 0), (0, \pi), (\pi, \pi)$ [13,15]. Assuming that e is the excitation of lowest energy, the low-energy edge of the continuum of excited states will reveal the fractionalization class of e in the dynamical structure factor of any operator that excites anyons. This is the main idea for detecting such SET states that we pursue in this Rapid Communication. However, we access the low-energy edge of the excited states via the information contained in the ground state instead of the excitation spectrum due to a nice property given by the PEPS, which we will introduce below.

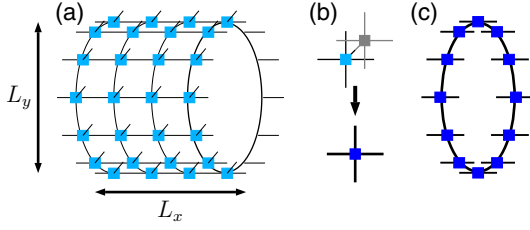


FIG. 1. (Color online) (a) Schematic representation of a PEPS on a cylinder of size $L_x \times L_y$. (b) When multiplying bra and ket of PEPS, we sum over the physical degrees of freedom, group the virtual indices of a bond, and arrive at the double tensor of a single site. (c) Placing L_y double tensors on a ring and tracing out the virtual degrees on the shared bonds, we form the transfer matrix of the cylinder.

PEPS and transfer matrix. PEPS is an ansatz that represents the wave function by locally entangled virtual pairs and a projector that maps the virtual system to the physical one [16]. It captures a wide range of phases including many with topological order [17–24]. A one-dimensional version, the MPS [25], has been used to classify phases of symmetric, gapped spin systems based on the projective representation of the symmetry group [26–28]. Here, we propose a method based on PEPS for the spectra of correlation lengths (SCL) of the system, which allows us to distinguish SET phases described by simple PEPSs.

In one-dimensional gapped systems, properties of the ground-state wave function (the MPS) are completely determined by the transfer matrix if translation symmetry is present. The spectrum of correlation lengths, which is given by the negative of the logarithm of (normalized) eigenvalues (λ) of the transfer matrix [25], is intuitively related to the spectrum of excitations made by all possible local operators with momentum quantum number k_x , $\lambda = |\lambda|e^{ik_x}$, at the minima of the spectrum [29].

The SCL can be generalized in two dimensions; however the momentum quantum number in the y direction k_y is rigorously well defined as compared to the k_x estimated from the complex phases of the transfer matrix eigenvalues, as in the one-dimensional case mentioned above. Consider a cylindrical geometry (as in Fig. 1); we can define the SCL of the transfer matrix of a cylinder. If translation symmetry in the y direction is present, eigenvectors of the transfer matrix have well-defined momentum quantum numbers k_y , and the minimum of the SCL is given by

$$\epsilon_{(k_x, k_y)} = -\ln(|\lambda_{(k_x, k_y)}^{\max}|/\lambda_0), \quad (2)$$

where $\lambda_{(k_x, k_y)}^{\max} = e^{ik_x} |\lambda_{(k_x, k_y)}^{\max}|$ is the leading eigenvalue of the transfer matrix with momentum k_y excluding ground states λ_0 s (the largest eigenvalue among all sectors); in general it is a complex number with a phase e^{ik_x} where k_x is its momentum in the x direction [29]. We conjecture that $\epsilon_{(k_x, k_y)}$ at a given (k_x, k_y) are analogous to the low-energy edge of the two-anyon scattering continuum described earlier. Thus, we propose that the minima of the SCL can be used to distinguish SET phases, by analogy with the dynamic structure factor. Next we will test this conjecture by examining the minima of the SCL of the transfer matrix.

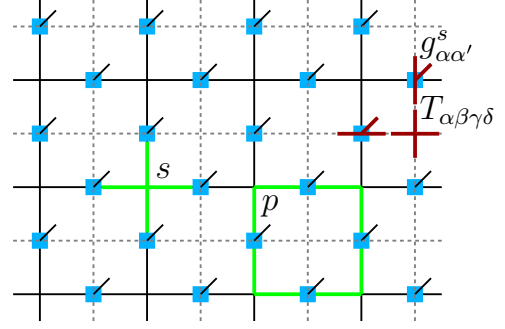


FIG. 2. (Color online) A demonstration of the PEPS wave function of the toric code model in the σ^z basis. Green lines denote the star operator A_s and the plaquette operator B_p , and dark red lines denote the position of the 4-index T tensor and the 3-index g tensor, which are placed at the vertices and bonds of the dual lattice (denoted in dashed lines).

Modified toric code model. To construct the simplest \mathbb{Z}_2 spin liquid which realizes all possible projective quantum numbers of the translation symmetry, we consider the toric code Hamiltonian on a square lattice

$$H = -K_e \sum_s A_s - K_m \sum_p B_p, \quad (3)$$

where $A_s = \prod_{l \in s} \sigma_l^x$ is defined on the vertex s and $B_p = \prod_{l \in p} \sigma_l^z$ is defined on the plaquette p , and the sum runs over all vertices and all plaquettes. This Hamiltonian has four degenerate ground states [30]. Depending on the signs of K_m and K_e , the quasiparticles e , m , if created, will move in a background of 0 or π flux. That is, $\eta_e = \text{sgn} K_m$, $\eta_m = \text{sgn} K_e$ in Eq. (1). The ground states on a torus can be simply represented by PEPSs of bond dimension $D = 2$. We now describe the PEPSs for all choices of $K_e = \pm 1$ and $K_m = \pm 1$.

The PEPSs are composed of 4-index tensors $T_{\alpha\beta\gamma\delta}$ at the vertices and 3-index tensors $g_{\alpha\alpha'}$ at the bonds of the direct (or dual) lattice, where s represents physical degrees of freedom and Greek letters represent virtual ones. Whether we take the direct or dual lattice depends on the choice of using a local σ^x or σ^z basis. Figure 2 represents a PEPS defined in the σ^z basis, where the tensor $T_{\alpha\beta\gamma\delta}$ is placed at the vertices of the dual lattice; this is the representation we choose throughout this paper unless specified otherwise. The virtual index runs from 0 to 1, where 0 means $|\uparrow\rangle$ and 1 means $|\downarrow\rangle$. The wave function has the form $|\psi\rangle = \text{Tr}\{T^{\otimes V} g^{\otimes B}\}$, where V means all vertices and B means all bonds, and the trace is taken over all common virtual degrees. The T tensor is

$$T_{\alpha\beta\gamma\delta} = \begin{cases} 1, (0,0) & (\alpha + \beta + \gamma + \delta) \% 2 = 0, \\ 0, (1,1) & (\alpha + \beta + \gamma + \delta) \% 2 = 1, \end{cases} \quad (4)$$

which, together with the condition that the only nonzero elements of the g tensor are g_{ss}^s (see below), enforces the condition $B_p|\psi\rangle = +(-)|\psi\rangle$, corresponding to $K_m > 0 (K_m < 0)$. The elements of the g tensor are

$$g_{\alpha\alpha'}^s = \begin{cases} a, & \alpha = \alpha' = s = 0, \\ 1, & \alpha = \alpha' = s = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

where a is some number which can also depend on the position of the bond. We ask that the wave function satisfies $A_s|\psi\rangle = +(-)|\psi\rangle$ for $K_e > 0$ ($K_e < 0$); this is nothing but asking that the amplitudes for configurations related by flipping the four spins on the vertex s differ by $+(-)$. If $K_e > 0$ the solution is obvious: $a = 1$ on all bonds. However, if $K_e < 0$, it requires one and only one a on each plaquette of the dual lattice to be -1 , in which case one has to break the lattice translation symmetry in one direction in order to keep the bond dimension $D = 2$ (Supplemental Material [31]).

The excitations of the Hamiltonian Eq. (3) have no dynamics, which corresponds in the ground state $|\psi\rangle$ to the fact that all spin configurations satisfying Eqs. (4) and (5) have equal weight in magnitude. To create some dynamics without increasing the bond dimension of the tensor, we put a local diagonal operator $\text{diag}(w, 1)$ on each physical spin with $0 \leq w \leq 1$, which corresponds to a ground state of the Hamiltonian [32,33]

$$\begin{aligned} H' &= H + K_e \sum_s w^{-\sum_{l \in s} \sigma_l^z} \\ &\approx H + h \sum_l \sigma_l^z + \text{const.} \end{aligned} \quad (6)$$

Here, the Zeeman field $h = -2K_e \ln w$ is a good description of the perturbed state for $h \ll K_e$, or $w \approx 1$. Throughout this Rapid Communication, we use $w = 0.9$, which corresponds to $h \ll |K_m|, |K_e|$ and is deep in the topologically ordered phase. For the phase diagram of the modified PEPS as a function of w in the case of $K_m < 0$ and $K_e > 0$, see the Supplemental Material [31].

Numerical detection of the projective quantum number under translation symmetries. We take the PEPS wave functions described above and calculate the SCL of the transfer matrix [see Fig. 1(b)] for different signs of K_m and K_e . The transfer matrix of a cylinder with translation symmetry T_y can be block-diagonalized in the momentum basis, which means that each left and right eigenvector has a well-defined momentum quantum number k_y . We start by explicitly writing the transfer matrix in real space into a block-diagonal form in momentum space (Supplemental Material [31]). Once we have obtained the transfer matrix in the momentum basis, we diagonalize each block with momentum $k_y = 2\pi m/L_y$, $m = 0, 1, \dots, L_y - 1$, find the minima of the normalized SCL $\epsilon_{(k_x, k_y)} = -\ln(|\lambda_{(k_x, k_y)}^{\max}|/\lambda_0)$, and plot $\epsilon_{(k_x, k_y)}$ as a function of k_y for $k_x = 0$ and π . Figure 3 illustrates the minima of the SCL for $K_m > 0$ and $K_e > 0$, in which case, all eigenvalues of the transfer matrix are real and positive. The results for $K_m < 0$, $K_e > 0$, and L_y even are presented in Fig. 4: $\epsilon_{(0, k_y)}$ corresponds to the largest real and positive eigenvalue λ at each k_y excluding the two degenerate λ_0 s at $k_y = 0$, while $\epsilon_{(\pi, k_y)}$ corresponds to the smallest real and negative eigenvalue λ at each k_y . In the case of odd L_y for $K_m < 0$ and $K_e > 0$, matrices at each k_y further form into two noncommuting blocks, encoding the fact that applying the transfer matrix flips the eigenvalue of $\prod_l \sigma_l^z$, where the product is over a loop encircling the system in the y direction. This corresponds to a breaking of translation symmetry (T_x) in a one-dimensional limit ($L_x \rightarrow \infty$ with L_y fixed). All eigenvalues of the transfer matrix come in \pm pairs; thus $\epsilon_{(0/\pi, k_y)}$ are identical, as presented in Fig. 5. We find

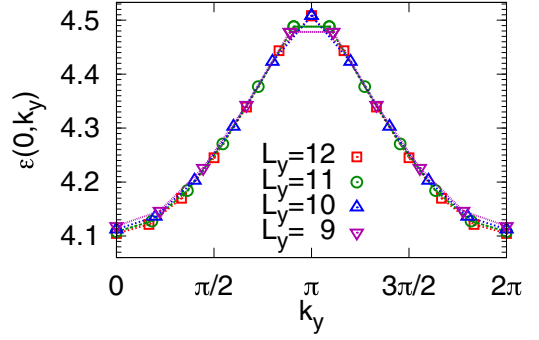


FIG. 3. (Color online) The minima of SCL at momentum k_y for coupling $K_m > 0$ and $K_e > 0$ using PEPS at $w = 0.9$. Since all eigenvalues of the transfer matrix are real and positive, only $\epsilon_{(0, k_y)}$ is presented.

that the minima of the SCL are indeed doubly periodic when $K_m < 0$, but not when $K_m > 0$.

For the case of $K_e < 0$ and $K_m < 0$ ($K_m > 0$), the resulting $\epsilon_{(0, k_y)}$ are exactly the same as above, because the low-energy excitations are dictated by coupling K_m regardless of the sign of K_e . Note that when $K_e < 0$, $\epsilon_{(\pi, k_y)}$ is not accessible, because

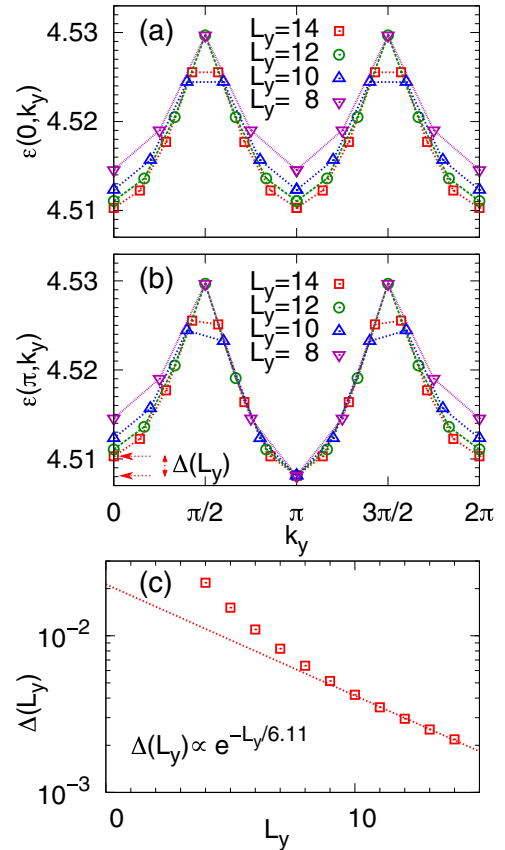


FIG. 4. (Color online) The minima of SCL at momentum k_y for coupling $K_m < 0$ and $K_e > 0$ using PEPS at $w = 0.9$ for even L_y . (a) $\epsilon_{(0, k_y)}$ corresponds to the largest real and positive λ in sector k_y except λ_0 s. (b) $\epsilon_{(\pi, k_y)}$ corresponds to the smallest real and negative λ in sector k_y . (c) The splitting $\Delta(L_y) = \epsilon_{(\pi, 0)} - \epsilon_{(\pi, \pi)}$ as illustrated in (b) vanishes exponentially as a function of length L_y .

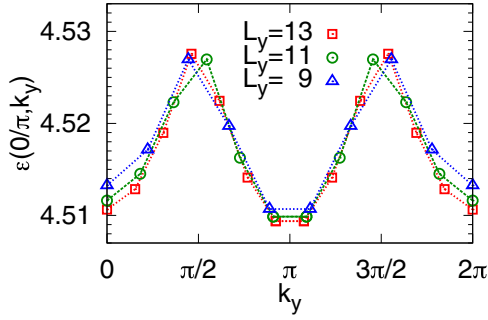


FIG. 5. (Color online) The minima of SCL at momentum k_y for coupling $K_m < 0$ and $K_e > 0$ using PEPS at $w = 0.9$ for odd length L_y . Since eigenvalues including the largest come in \pm pairs, $\epsilon_{(0,k_y)}$ and $\epsilon_{(\pi,k_y)}$ are identical.

the transfer matrix at any L_y consists of two adjacent columns of the lattice; thus all eigenvalues are positive.

If we want the projective quantum number of m particles, we can take a perturbed Hamiltonian as

$$H'' = -K_e \sum_s A_s - K_m \sum_p B_p + h' \sum_l \sigma_l^x \quad (7)$$

instead, and correspondingly choose a PEPS representation defined in the σ^x basis, where acting with a local operator $\text{diag}(w, 1)$ is equivalent to adding a perturbation $\delta H = h' \sum_l \sigma_l^x$. The low-energy excitations are two m particles and the sign of K_e determines whether the m particle hops in a background of 0 or π flux on the dual lattice. Once the projective quantum numbers η_e and η_m are known, one completely determines the SET class of the state if the symmetry group consists only of translation [2].

Conclusion. We have presented a numerical method to distinguish symmetry-enriched topological (SET) phases with

fractionalized translation symmetry. This method uses the spectrum of correlation lengths (SCL) of the transfer matrix on a cylinder to represent qualitatively the excitation spectrum of the system as a function of k_y for special $k_x = 0, \pi$, if the ground-state wave function is available in terms of a projected entangled pair state (PEPS). From the fact that the nontrivial fractional quantum numbers of quasiparticles under translation symmetries will be manifest as enhanced Brillouin zone periodicity in the dispersion relation, one can read out the projective quantum number of the low-energy quasiparticles from the behavior of the minima of the SCL. We benchmarked this method with the toric code model under perturbation. Modifying the sign of the coupling coefficients in front of operator A_s and B_p and the perturbation terms, we were able to generate topologically ordered ground states with preferred e -particle or m -particle low-energy excitations in a background of either 0 or π magnetic flux, which realizes all symmetry classes with this topological order and symmetry [2]. We expressed the ground states of the modified toric code Hamiltonians as bond dimension $D = 2$ PEPSs and calculated the minima of the SCL of the transfer matrix on a cylinder, and the pattern of SCL revealed the fractional quantum numbers of the low-energy quasiparticles under translation symmetries.

This method can be generalized to detect projective quantum numbers of SET phases under a broader symmetry group including translations and other space-group symmetries.

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